No-go theorem for boson condensation in topologically ordered quantum liquids

2016 New J. Phys. 18 123009

(http://iopscience.iop.org/1367-2630/18/12/123009)

New Journal of Physics
The open access journal at the forefront of physics

Deutsche Physikalische Gesellschaft DPG
IOP Institute of Physics

Published in partnership with: Deutsche Physikalische Gesellschaft and the Institute of Physics

CrossMark

OPEN ACCESS

RECEIVED
8 September 2016

REVISED
17 November 2016

ACCEPTED FOR PUBLICATION
21 November 2016

PUBLISHED
7 December 2016

PAPER

No-go theorem for boson condensation in topologically ordered quantum liquids

Titus Neupert[1,2,5], Huan He[3], Curt von Keyserlingk[2], Germán Sierra[4] and B Andrei Bernevig[3]

1 Department of Physics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland
2 Princeton Center for Theoretical Science, Princeton University, Princeton, NJ 08544, USA
3 Department of Physics, Princeton University, Princeton, NJ 08544, USA
4 Instituto de Física Teórica, UAM-CSIC, Madrid, Spain
5 Author to whom any correspondence should be addressed.

E-mail: titus.neupert@uzh.ch

Keywords: topological order, Bose–Einstein condensation, topological quantum field theory

Abstract

Certain phase transitions between topological quantum field theories (TQFTs) are driven by the condensation of bosonic anyons. However, as bosons in a TQFT are themselves nontrivial collective excitations, there can be topological obstructions that prevent them from condensing. Here we formulate such an obstruction in the form of a no-go theorem. We use it to show that no condensation is possible in $SO(3)_k$ TQFTs with odd k. We further show that a 'layered' theory obtained by tensoring $SO(3)_k$ TQFT with itself any integer number of times does not admit condensation transitions either. This includes (as the case $k = 3$) the noncondensability of any number of layers of the Fibonacci TQFT.

Topological order, a fundamental concept in quantum many-body physics, is best understood in two-dimensional gapped quantum liquids, such as the fractional quantum Hall effect and certain spin liquids [1–9]. In these systems, quasiparticle excitations with anyonic quantum-statistical properties emerge [10]. Their fusion and braiding behavior at large distances define a topological quantum field theory (TQFT), which characterizes the universal properties of the phase [11–14].

The phase transitions between topological phases are, most of the times, driven by the condensation of bosons [11, 15–25]. In the context of TQFTs, a boson is an emergent quasiparticle in the topologically ordered phase with bosonic self-statistics, but which could have nontrivial fusion and braiding relations with the other anyons. Such a quasiparticle can potentially undergo Bose–Einstein condensation, causing a phase transition to another topologically ordered phase. The topological data of the new phase can be inferred from those of the initial topological order [25].

One motivation to study condensation transitions is to classify topological order. An important example are the 16 types of gauged chiral superconductors introduced by Kitaev [3]. Kitaev showed that while two-dimensional superconductors are classified by an integer $\mathbb{Z}$, only 16 bulk phases are topologically distinct. This construction can be understood by considering ℓ layers of initially disconnected chiral p-wave superconductors, i.e., elementary (Ising) TQFTs. Upon introducing generic couplings between these layers, one obtains a single layer of a chiral ℓ-wave superconductor, which corresponds to a specific TQFT in Kitaev's classification. This physical process of coupling the layers (by condensing inter-layer cooper pairs), corresponds to a condensation transition on the level of the TQFTs. For every $\ell < 16$, there is a unique condensation possible and one obtains exactly 16 distinct TQFTs including Ising, the toric code and the double semion model. They determine the nature of the topologically protected excitations in the vortices of each superconductor, including their braiding statistics. In essence, this $\mathbb{Z}_{16}$ classification can be seen as a property of the Ising TQFT.

It is imperative to ask whether multi-layer systems of other TQFTs show a similar collapse of the classification from $\mathbb{Z}$ to $\mathbb{Z}_N$ for some integer N. In this paper, we derive a criterion for when this is not the case, i.e., when the $\mathbb{Z}$ classification generated by a given TQFT is stable. This criterion is based on the fact that there exist bosonic anyons that cannot be condensed. An example are the bosons in multi-layered Fibonacci topological order [17, 25, 26]. In this work, we generalize this observation by formulating a no-go theorem that constitutes a sufficient obstruction against the condensation of a boson. Our criterion and its proof are given using the tensor category formulation of topological order [3, 27–34], which we can use to describe the condensation transition axiomatically [16, 17, 25]. We apply our no-go theorem to several examples, including the forementioned multi-layer Fibonacci TQFTs.

Formalism—We use the algebraic formulation of anyon condensation discussed in [25]. Here we simply restate the important relations and refer the reader to [25] for details. A *fusion category* is characterized by a set of anyons $a,\ b,\ c,\ \ldots$ and fusion rules $a\times b=\sum_c N_{ab}^c c$ between them. The quantum dimension d_a gives the size of the nonlocal internal Hilbert space associated with anyon a, and is equal to largest eigenvalue of the matrix N_a with elements $(N_a)_{bc}\equiv N_{ab}^c$. A *braided tensor category* has additional structure, of which we will use the topological spin θ_a of a, a complex number with $|\theta_a|=1$. Bosons are defined by $\theta_a=1$. A special role is played by the vacuum anyon as the unique identity element of fusion. It is a boson with quantum dimension 1.

Condensation is based on a mapping, called restriction, between the anyons a in the original TQFT $\mathcal{A}$ and the anyons t in the condensed fusion category $\mathcal{T}$ characterized by integers $n_a^t\in\mathbb{Z}_{\geqslant 0}$:

$$a\mapsto a^{\downharpoonleft}\equiv\sum_{t\in\mathcal{T}}n_a^t t,\qquad\forall\, a\in\mathcal{A}.\tag{1}$$

If more than one particle appears on the right-hand side of equation (1), we say that the a particle *splits*. If $n_a^t\neq 0$, we say t is in the *restriction* of a or $t\in a^{\downharpoonleft}$. We require that $n_1^{\varphi}=1$, where φ and 1 are the vacua in $\mathcal{T}$ and $\mathcal{A}$, respectively. Imposing that condensation commutes with fusion implies the fundamental relation [25]

$$\sum_{r,s\in\mathcal{T}}n_a^r n_b^s\tilde{N}_{rs}^t=\sum_{c\in\mathcal{A}}N_{ab}^c n_c^t,\tag{2}$$

between the fusion coefficients N_{ab}^c in $\mathcal{A}$ and the fusion coefficients $\tilde{N}_{rs}^t$ in $\mathcal{T}$. A corollary to equation (2) [25] is

$$d_a=\sum_{r\in\mathcal{T}}n_a^r d_r,\qquad\forall\ a\ \in\mathcal{A}.\tag{3}$$

The restriction is compatible with conjugation to antiparticles, i.e., $n_a^t=n_{\bar{a}}^{\bar{t}}$, where bar denotes the (unique) antiparticle of an anyon. We say particle *a condenses* if $\varphi\in a^{\downharpoonleft}$, i.e., $n_a^{\varphi}\neq 0$. Common knowledge in condensed matter physics says that any bosons can condense. However, it may also occur that a specific boson a cannot condense, i.e., there is no solution to the above equations with $n_a^{\varphi}\neq 0$. This is the situation we shall analyze in this paper.

Finally, the following definition is useful for formulating our no-go theorem: for a given anyon b, a subset $\mathcal{I}_b=\{a_1,\ldots,a_m\}$ of anyons is called a *set of zero modes localized by b* [35] if for all $i,\ j=1,\ldots,m$:

(1) The fusion products $a_i\times a_j$ do not contain condensable bosons, except the identity if $a_i=\bar{a}_j$,[6]

(2) all a_i are zero modes of b, by which we mean $a_i\times b=b+\cdots$, (i.e. $N_{a_i b}^b>0$)

(3) if a particle a_i is in $\mathcal{I}_b$ then so is its antiparticle.

Note that the choice of $\mathcal{I}_b$ for a given boson b is not unique and that $\mathcal{I}_b$ may or may not contain the identity. (The above conditions are satisfied in both cases.) Typically, we will be interest to find a set $\mathcal{I}_b$ that is as large as possible. To motivate the terminology of the set $\mathcal{I}_b$, observe that $N_{ab}^b>0$ implies that a anyons can always be emitted or absorbed by b. Therefore, b must carry a zero-mode excitation of a. We can now state our first main result, a general condition under which a boson B cannot condense. It is an obstruction that is sufficient to show that condensation of B cannot occur.

No-go theorem—A boson B cannot condense if there exists a set $\mathcal{I}_B$, such that the sum of the quantum dimensions of all anyons in $\mathcal{I}_B$ exceeds the quantum dimension of B, i.e., if

$$d_B<d_{a_1}+d_{a_2}\ +\ \cdots\ +\ d_{a_m}.\tag{4}$$

Proof. We start by showing that all particles in $\mathcal{I}_B$ do not split, and have distinct restrictions. This follows from inspection of equation (2) for $t=\varphi$, $a=a_i$, $b=\bar{a}_j$,

[6] In demanding that $a_i\times a_j$ does not contain condensable bosons, as opposed to not containing any bosons at all (except the identity), we are anticipating a inductive application of the no-go theorem. Once we have shown that a boson B, whose set $\mathcal{I}_B$ is such that $a_i\times a_j$, with $a_i,\ a_j\in\mathcal{I}_B$, does not contain any boson (except the identity), is uncondensable, it is allowed that B appears in the fusion product $a_i\times a_j$ of the set $\mathcal{I}_{B'}$ of another boson B'.

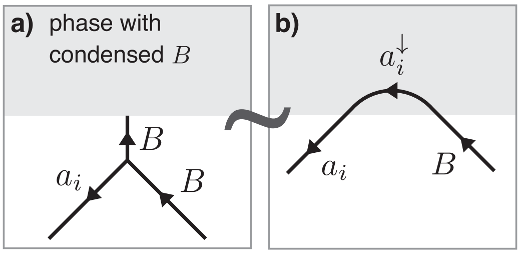

Figure 1. Tunneling processes mediated by an anyon condensate. The gray region is a phase in which a boson B is condensed. (a) Vertex of a boson B that localizes a zero mode of anyon a_i. In the condensed phase, B can be converted into an identity particle world line (not shown). By the axioms of anyon condensation, processes (a) and (b) are equivalent, i.e., B can be converted into a_i by tunneling through the condensate.

$$\sum_{r\in\mathcal{T}} n_{a_i}^r n_{a_j}^r = \delta_{i,j} + \sum_{c\neq 1} N_{a_i\bar{a}_j}^c n_c^{\varphi}, \tag{5}$$

where we used $n_c^{\varphi} = n_{\bar{c}}^{\varphi}$ and $n_{\bar{a}_j}^{\bar{r}} = n_{a_j}^r$. By assumption, there are no condensable bosons in $a_i \times \bar{a}_j$, hence $N_{a_i\bar{a}_j}^c$ and n_c^{φ} cannot be both nonzero for any $c \neq 1$. Thus $\sum_r n_{a_i}^r n_{a_i}^r = 1$, implying a single restriction $a_i^{\downarrow}$ of a_i, with $d_{a_i^{\downarrow}} = d_{a_i}$ using equation (3). Moreover, $\sum_r n_{a_i}^r n_{a_j}^r = 0$ if $i \neq j$, implying that the restrictions of $a_i \neq a_j$ are distinct particles.

With this knowledge about the restrictions of the a_i, equation (2) for $t = \varphi$, $a = a_i$, $b = \bar{B}$ evaluates to

$$n_{\bar{B}}^{\bar{a}_i^{\downarrow}} = n_B^{a_i^{\downarrow}} = \sum_c N_{a_i\bar{B}}^{\bar{c}} n_c^{\varphi} \geqslant N_{a_iB}^B n_B^{\varphi}, \tag{6}$$

where we used $N_{a_i\bar{B}}^{\bar{B}} = N_{a_iB}^B$ Inserting this inequality in equation (3) for $a = B$, and using $d_{a_i} = d_{a_i^{\downarrow}}$, we have

$$d_B \geqslant n_B^{\varphi} \sum_{i=1}^{m} N_{a_iB}^B d_{a_i}. \tag{7}$$

It follows that in a situation where equation (4) holds, equation (7) implies $n_B^{\varphi} = 0$, i.e., B does not condense. (Note that in the case $N_{a_iB}^B > 1$, a stronger form of equation (4) with d_{a_i} is replaced by $N_{a_iB}^B d_{a_i}$ holds.)

To follow up with a pictorial representation of these equations, consider the tunneling of anyons across the domain wall as shown in figure 1, where each particle a in the uncondensed theory is converted into its restriction $a^{\downarrow}$ in the gray region. Figure 1(a) shows a vertex allowed by the fusion rule $a_i \times B \to B$ in the uncondensed phase. The boson B enters the condensed phase, where it can disappear as it is part of the condensate (one of its restrictions is the vacuum φ, the world lines of which can be removed at will). By the fundamental assumption that fusion and condensation commute (which is at the heart of equation (2)), figure 1(a) is equivalent to figure 1(b). The latter represents a coherent tunneling process that is mediated by the condensate and converts B into any of the a_i. The existence of this process implies that the distinct restriction $a_i^{\downarrow}$ of any a_i must be in the restriction of B. Hence, by equation (3), the quantum dimension of B must be large enough to accommodate all the distinct restrictions of the a_i, if B condenses. Therefore if we find sufficiently many a_i such that equation (4) holds, B cannot condense. □

Note that the no-go theorem does not *a priori* require knowing the braiding data of $\mathcal{A}$—although the modular tensor category structure fixes that data to some extend. The theorem involves only data obtainable from N_{ab}^c. We remark that the no-go theorem can only ever yield an obstruction against the condensation of non-Abelian bosons. For Abelian bosons, the theory after condensation can be constructed explicitly, which is a constructive proof that there is no obstruction [25].

We now demonstrate that the no-go theorem is practically useful by considering three examples: (i) multiple layers of the Fibonacci TQFT, (ii) single layers of the $SO(3)_k$ TQFT for k odd, and (iii) multiple layers of the latter. We will show that all these theories, while containing bosons, do not admit condensation transitions. All the bosons are noncondensable. Additional general results, concerning for instance TQFTs with a condensing Abelian sector and with only a single boson, are given in appendix A.

Example (i): Multiple layers of Fibonacci—The Fibonacci category $\mathcal{A}_{\mathrm{Fib}}$ is a non-Abelian TQFT containing just one nontrivial particle τ with a fusion rule $\tau \times \tau = 1 + \tau$, a topological spin $\theta_\tau = \mathrm{e}^{\mathrm{i}4\pi/5}$, and a quantum dimension $d_\tau = \phi$ given by the golden ratio $\phi = (1 + \sqrt{5})/2$. As $\mathcal{A}_{\mathrm{Fib}}$ does not contain any nontrivial boson, it cannot undergo a condensation transition. We are interested whether the TQFT formed by N identical layers of $\mathcal{A}_{\mathrm{Fib}}$, i.e., the TQFT $\mathcal{A}_{\mathrm{Fib}}^{\otimes N}$, admits a condensation transition. The TQFT $\mathcal{A}_{\mathrm{Fib}}^{\otimes N}$ contains 2^N particles corresponding to all possible distributions of τ-particles over the N layers. For each $r = 0, \ldots, N$ there are $\binom{N}{r}$ so-called $(r\tau)$ particles with τ's in exactly r layers, each with spin $\theta_{(r\tau)} = \mathrm{e}^{\mathrm{i}4\pi r/5}$ and quantum dimension

$d_{(r\tau)} = \phi^r$. The unique $r = 0$ particle is the identity of $\mathcal{A}_{\mathrm{Fib}}^{\otimes N}$. From the topological spin, the bosons in $\mathcal{A}_{\mathrm{Fib}}^{\otimes N}$ are $(r\tau)$ particles with $r = 5n,\ n \in \mathbb{Z}$. Using the no-go theorem, we show that none of these bosons can condense.

Using proof by induction on $n \geqslant 1$, we show that for any $(5n\tau)$ boson B, there exists a set $\mathcal{I}_{(5n\tau)}$ such that equation (4) holds. We first consider the case $n = 1$. Given a (5τ) boson, we must construct a set $\mathcal{I}_{(5\tau)}$ for this boson. Consider the set formed by all (2τ) particles obtained by replacing any 3τ's in the boson with a 1. There are $\binom{5}{2} = 10$ such (2τ) particles for a given (5τ) boson. They form a set $\mathcal{I}_{(5\tau)}$ that obeys point 1–3 from the definition: point 1 holds as any product of two of these particles has at most 4 τs and is therefore not a (potentially condensable) boson. Points 2 and 3 can be checked by using the Fibonacci fusion rules in each layer. Finally, equation (4) holds because

$$d_{(5\tau)} = \phi^5 < 10\phi^2 = \sum_{a_i \in \mathcal{I}_{(5\tau)}} d_{a_i} \tag{8}$$

evaluates to about $11.1 < 26.2$. We conclude that none of the (5τ) bosons condense for any number N of layers of Fibonacci TQFT.

For the induction step, we assume that none of the $(5n\tau)$ bosons can condense for $n < n_0,\ n_0 > 1$, and we show that the same holds for the $(5n_0\tau)$ bosons. Define $r_0 := \lfloor (5n_0 - 1)/2 \rfloor$, where $\lfloor x \rfloor$ is the largest integer smaller than or equal to x. For a given $(5n_0\tau)$ boson, form the set $\mathcal{I}_{(5n_0\tau)}$ out of all $(r_0\tau)$-particles that are obtained by replacing any $(5n_0 - r_0)\tau$'s in the boson $(5n_0\tau)$ with a 1. There are $\binom{5n_0}{r_0}$ such $(r_0\tau)$ particles. They form a set $\mathcal{I}_{(5n_0\tau)}$ for $(5n_0\tau)$. In particular their fusion products can only contain $(5n\tau)$-bosons with $n < n_0$, which cannot condense by assumption. Equation (4) reads for this case

$$\phi^{5n_0} < \binom{5n_0}{r_0} \phi^{5n_0 - r_0}. \tag{9}$$

Using that $r_0 \sim 5n_0/2$ and $\binom{5n_0}{5n_0/2} \sim 4^{5n_0/2} \Big/ \sqrt{\pi 5n_0/2}$ for large n_0, we obtain that the right-hand side of equation (9) grows like $4^{5n_0/2}\phi^{5n_0/2}/\sqrt{n_0}$, asymptotically dominating the left-hand side. An explicit evaluation yields that equation (9) holds for any $n_0 \geqslant 1$ in fact. We have thus shown that none of the $(5n_0\tau)$ bosons can condense. This concludes the induction step and the proof that no boson in $\mathcal{A}_{\mathrm{Fib}}^{\otimes N}$ can condense.

Example (ii): Single layer of $\mathrm{SO}(3)_k$—Our second example focuses on the (single-layer) TQFTs associated with the Lie group SO(3) at values of odd level k. They contain bosons for an infinite subset of k. We show that none of these bosons can condense. The $\mathrm{SO}(3)_k$ TQFTs with k odd have $(k + 1)/2$ anyons $j = 0, \cdots, (k - 1)/2$ with

$$d_j = \frac{\sin\left(\pi \frac{2j+1}{k+2}\right)}{\sin[\pi/(k+2)]}, \qquad \theta_j = \mathrm{e}^{2\pi \mathrm{i} j \frac{j+1}{k+2}}. \tag{10}$$

We note that for k odd, all particles have distinct quantum dimensions. The fusion rules are

$$N_{j_1 j_2}^{j_3} = \begin{cases} 1 & |j_1 - j_2| \leqslant j_3 \leqslant \min\{j_1 + j_2,\ k - j_1 - j_2\} \\ 0 & \text{else} \end{cases}. \tag{11}$$

The smallest odd k for which $\mathrm{SO}(3)_k$ contains a boson is $k = 13$, in which $j = 5$ is a boson—an uncondensable one, as we shall see.

The topological spins θ_j yield the condition $j(j + 1) = k + 2$ for the lowest j that may correspond to a boson (aside from the vacuum $j = 0$). (Frequently, this condition cannot be met with integer j, as in the $k = 13$ example, and the lowest boson appears at even higher j.) We conclude that the first boson after $j = 0$ cannot occur for j lower than

$$j_0 = \lfloor \sqrt{k + 9/4} - 1/2 \rfloor. \tag{12}$$

We will now discuss separately bosons j in the three ranges (see figure 2 for two examples)

$$\text{I.} \quad j_0 \leqslant j \leqslant \lfloor k/4 \rfloor, \tag{13a}$$

$$\text{II.} \quad \lfloor k/4 \rfloor < j \leqslant \frac{k-1}{2} - \left\lfloor \frac{j_0 - 1}{2} \right\rfloor, \tag{13b}$$

$$\text{III.} \quad \frac{k-1}{2} - \left\lfloor \frac{j_0 - 1}{2} \right\rfloor < j \leqslant \frac{k-1}{2}. \tag{13c}$$

Due to equation (12), bosons j_B in range III have no bosons in their fusion product $j_B \times j_B$, other than the identity. Thus, from equation (2) for $t = \varphi$, and the fact that B are their own antiparticles, we conclude that they cannot split. Using equation (3) and the fact that they have $d_{j_B} > 1$, we conclude that they cannot restrict to the vacuum i.e., they cannot condense.

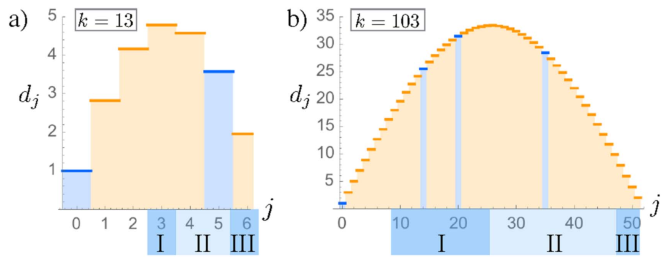

Figure 2. Quantum dimensions and bosons (blue columns) for $SO(3)_k$ theories with (a) $k = 13$ and (b) $k = 103$. These are the smallest k, for which $SO(3)_k$ contains two and four bosons, respectively. Indicated are also the ranges I–III defined in equation (13). The maximum quantum dimension coincides with the boundary between range I and II in equation (13). For instance, to apply the no-go theorem to the $j = 5$ boson in (a), choose $\mathcal{I}_{j=5} = \{j = 2\}$ and use that $d_5 \approx 3.6$ is smaller than $d_2 \approx 4.2$.

We now use our no-go theorem to show that bosons j_B in range I are noncondensable. Specifically, we show that the particles $0 < j < \lfloor j_B/2 \rfloor$ form a set $\mathcal{I}_{j_B}$ of j_B obeying equation (4). Before establishing that they satisfy the conditions for a set $\mathcal{I}_{j_B}$, let us show that equation (4) holds for $\mathcal{I}_{j_B}$. For large k, we can rely on the following asymptotic estimate. Using that the sine function in equation (10) is monotonously increasing with negative second derivative for $j \leqslant \lfloor k/4 \rfloor$, the estimate

$$2j_B + 1 < \sum_{j=1}^{\lfloor j_B/2 \rfloor - 1} (2j + 1) \tag{14}$$

implies equation (4) for j_B in range I. This inequality holds for all $j_B \geqslant 10$. Using equation (12) we conclude that it applies to all bosons in range I for $k \geqslant 109$. We verified explicitly that inequality (4) holds (using the exact values of the quantum dimensions) for all bosons in range I for $k < 109$. Finally, it is readily verified using equation (11) that $\mathcal{I}_{j_B}$ form a set of zero modes localized by j_B provided that all bosons with $j < j_B$ cannot condense. The proof then proceeds straightforwardly by induction.

We apply our no-go theorem successively to bosons j_B in range II in order of increasing j_B. Using the result that all bosons in range I are uncondensable, one verifies that the particles j with $1 \leqslant j \leqslant \min\{k - 2j_B, \lfloor j_B/2 \rfloor - 1\}$ form a set $\mathcal{I}_{j_B}$. As for range I, we can estimate the quantum dimensions. From the relation $\sin[\pi(2j_B + 1)/(k + 2)] = \sin[\pi(k - 2j_B + 1)/(k + 2)]$ we can estimate the quantum dimension of j_B using $\sin[\pi(2j_B + 1)/(k + 2)] < \pi(k - 2j_B + 1)/(k + 2)$. The quantum dimensions of the anyons in $\mathcal{I}_{j_B}$ are estimated as for range I with $\sin[\pi(2j + 1)/(k + 2)] < \pi(2j + 1)/(k + 2)$. Using these estimates we find that if

$$k - 2j_B + 1 < \sum_{j=1}^{\min\{k-2j_B, \lfloor j_B/2 \rfloor - 1\}} (2j + 1) \tag{15}$$

holds, equation (4) follows. In the case $k - 2j_B < \lfloor j_B/2 \rfloor - 1$, equation (15) reduces to $1 < (k - 2j_B)^2 + (k - 2j_B)$, which is true for all j_B in range II for all k. In the case $k - 2j_B > \lfloor j_B/2 \rfloor - 1$, equation (15) simplifies to $k + 2 < 2j_B + (\lfloor j_B/2 \rfloor)^2$, which holds for all j_B in range II if $k \geqslant 37$. We verified explicitly that equation (4) holds for all bosons in range II if $k < 37$ (they appear in $k = 13, 19, 31$). This concludes our proof that no condensation transition is possible in the $SO(3)_k$ TQFT for any odd k.

We note that this result can be readily extended to $SU(2)_k$ with k odd, since $SO(3)_k$ is the projection of $SU(2)_k$ to anyons with integer j. One simply includes the half-integer j anyons in the theory (none of which are bosons). The sets $\mathcal{I}_b$ as defined above remain the same and so do all the quantum dimensions. Hence, we also showed the noncondensability of $SU(2)_k$, with k odd. This is consistent with the ADE classification of $SU(2)_k$ [36]: there are no off-diagonal modular invariant partition functions for odd k in $SU(2)_k$ [37]. Thus, the no-go theorem provides a proof of this fact that is complementary to the ADE classification.

Example (iii): Multiple layers of $SO(3)_k$—We can show that any number of layers of $SO(3)_k$, with k odd, does not contain condensable bosons. Fixing k, the proof proceeds again by induction. As induction base, we proof that all multi-layer anyons with a nontrivial particle in only a single layer (and the identity anyon in the other $k - 1$ layers) cannot condense nor split. To show that, we can use the single-layer result from example (ii). For the induction step, we assume that for a fixed $k_0 < k$ all multi-layer anyons with nontrivial particles in l layers, $1 \leqslant l \leqslant k_0$, cannot condense and do not split. We can then show that the same holds for multilayer anyons with nontrivial particles in $k_0 + 1$ layers, completing the induction. The details of this proof are given in appendix B.

Summary—We have presented a generally applicable no-go theorem against the condensation of a topological boson and illustrated it with several examples. The proof of our theorem uses mostly the fusion (as compared to the braiding) information of the TQFT. We showed a connection between our results and the ADE classification of SU(2)$_k$ theories, indicating that the no-go theorem might be useful for the classification of modular invariant partition functions of conformal field theories more broadly [25]. It would be interesting to study, whether other obstructions against boson condensation exist or whether our no-go theorem actually constitutes a necessary condition. In all examples we know, noncondensability is captured by the no-go theorem.

The no-go theorem can be used to study whether a TQFT is $\mathbb{Z}_N$ graded under layering. This provides a way to classify TQFTs depending on whether N is finite or infinite. As a venue for future work, when restricting the condensations to those that preserve certain symmetries of the anyon model, one could similarly classify symmetry enriched topological phases, and with this also symmetry protected topological phases without intrinsic topological order. The classification of the latter is often related to the former upon gauging the protecting symmetry [38, 39].

Acknowledgments

The authors thank Parsa Bonderson for useful discussions. This work was supported by a Simons Investigator Award, ONR—N00014-14-1-0330, ARO MURI W911NF-12-1-0461, NSF-MRSEC DMR-1420541, NSF EAGER AWD1004957, Department of Energy DE-SC0016239, and the Packard Foundation.

Appendix A. No-go theorem with Abelian sector

We have seen from the examples discussed in the main text, that the no-go theorem can often be used to not only show that individual bosons in a TQFT cannot condense, but that an entire TQFT is not condensable. Here, we extend this discussion to examples of TQFTs that have noncondensable sub-structures. This problem is motived by physical examples: in the fractional quantum Hall effect, for instance, one frequently discusses phases that are described by a direct (or semi-direct) product of an Abelian and a non-Abelian TQFT. A simple example is the $\mathbb{Z}_3$ Read–Rezayi state of bosons, which is described by the TQFT $\mathcal{A}_{\mathrm{Fib}} \times \mathbb{Z}_2$. While such a theory admits condensations, already in the $\mathbb{Z}_2^{\otimes N}$ sector, when enough layers are considered, one has the intuition that the noncondensability of Fibonacci should still constrain the possible condensations.

Lemma 1. *Consider a TQFT*

$$\mathcal{A} \times \mathcal{X}, \tag{A1}$$

where $\mathcal{X}$ is an Abelian TQFT (i.e., all its anyons have quantum dimension 1). Further, for all particles $b \in \mathcal{A}$ (not only for the bosons), except for the vacuum, let there exist a set $\mathcal{I}_b = \{a_1, \ldots, a_m\}$ of zero modes of b, containing anyons from $\mathcal{A}$, such that the quantum dimensions satisfy

$$d_b \leqslant \sum_{i=1}^{m} d_{a_i}. \tag{A2}$$

Then, any possible condensation transition will lead to a theory of the form

$$\mathcal{A} \times \mathcal{Y}, \tag{A3}$$

where the Abelian TQFT $\mathcal{Y}$ can be obtained from $\mathcal{X}$ through a condensation.

Proof. This lemma follows almost directly from the no-go theorem. Let us denote a particle from $\mathcal{A} \times \mathcal{X}$ by the pair (b, x) where $b \in \mathcal{A}$ and $x \in \mathcal{X}$. If (b, x) is boson, we can show that it has to be an uncondensable one, except if $b = 1$. The set

$$\mathcal{I}_{(b,x)} = \{(a_1, x), \cdots, (a_m, x)\}, \tag{A4}$$

(where $a_1, \cdots, a_m$ form a set $\mathcal{I}_b$ of zero modes of b whose existence is guaranteed by assumption) satisfies all the conditions 1–3 form the definition of a set of (b, x) zero modes. Since x is an Abelian particle, $d_x = 1$ and equation (A2) directly implies that the sum of the quantum dimensions of the particles in $\mathcal{I}_{(b,x)}$ satisfies the inequality (4) from the main text. Hence, (b, x) cannot condense. In turn, this implies any condensable boson in $\mathcal{A} \times \mathcal{X}$ is of the form $(1, x)$. A condensate of this form is transparent to the anyons in $\mathcal{A}$ and will thus leave this sub-TQFT unaffected. It will only induce a condensation $\mathcal{X} \to \mathcal{Y}$, so that the final theory is of the from (A3). □

We return to the example of $\mathcal{A}_{\mathrm{Fib}} \times \mathbb{Z}_2$. Consider N layers of this theory, i.e., $\mathcal{A}_{\mathrm{Fib}}^{\otimes N} \times \mathbb{Z}_2^{\otimes N}$. This multi-layer TQFT satisfies all assumptions of lemma 1: for each anyon $b \in \mathcal{A}_{\mathrm{Fib}}^{\otimes N}$, a choice for the set $\mathcal{I}_b$ is given by $\mathcal{I}_b = \{1, b\}$. This is so because all possible bosons appearing in the fusion product of $b \times b$ are uncondensable by the no-go theorem and the sum of the quantum dimensions of $\mathcal{I}_b$, given by $1 + d_b$ is larger than d_b. We conclude that the $\mathcal{A}_{\text{Fib}}^{\otimes N}$ structure is preserved under any condensation transition in such a theory.

Appendix B. Proof for example (iii), multiple layers of $\mathrm{SO}(3)_k$

In this section, we show that no condensation is possible in the TQFT $\mathrm{SO}(3)_k^{\otimes N}$ comprised of N layers of $\mathrm{SO}(3)_k$ for any odd k and any integer N. The proof goes by induction. We denote the particles in $\mathrm{SO}(3)_k^{\otimes N}$ with a shorthand notation. An anyon that has the identity particle from $\mathrm{SO}(3)_k$ in all layers, except for the k_0 layers $i_1, i_2, \cdots, i_{k_0}$, is denoted by $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0}}\}$. Here $1 \leqslant j_{i_l} \leqslant (k-1)/2$ can stand for any anyon from $\mathrm{SO}(3)_k$ (except the identity 0), for all $l = 1, \cdots, k_0$.

B.1. Induction base

First, consider particles $\{j_i\}$ with just one nontrivial anyon in some layer i. This will serve as the induction base. By the no-go theorem and our proof in example (ii), we know that no bosons of form $\{j_i\}$ can condense. (Use the particles with only one nontrivial in that same layer i to build the set $\mathcal{I}_{j_i}$ as elaborated for example (ii).) As a corollary, the anyons $\{j_i\}$ do not split: when fused with themselves no condensable boson appears in the fusion product, which prevents splitting by equation (2) from the main text for $t = \varphi$.

B.2. Induction step

We assume that for any $1 \leqslant l \leqslant k_0$ all $\{j_{i_1} j_{i_2} \cdots j_{i_l}\}$

(1) do not condense and

(2) do not split.

We now show the induction step, namely that all particles with nontrivial anyons in $(k_0 + 1)$ layers $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0+1}}\}$ neither condense nor split.

We begin by showing that $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0+1}}\}$ cannot condense. The particles $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0+1}}\}$ can be obtained by fusing a $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0}}\}$ with a $\{j_{i_{k_0+1}}\}$, where $i_{k_0+1} \notin \{i_1, \cdots, i_{k_0}\}$. In this case, equation (2) from the main text reads for $t = \varphi$

$$\tilde{N}^{\varphi}_{\{j_{i_1} \cdots j_{i_{k_0}}\}^{\downarrow}, \{j_{i_{k_0+1}}\}^{\downarrow}} = n^{\varphi}_{\{j_{i_1} j_{i_2} \cdots j_{i_{k_0}} j_{i_{k_0+1}}\}}. \tag{B1}$$

Now, because of the uniqueness of the antiparticle, $\tilde{N}^{\varphi}_{\{j_{i_1} \cdots j_{i_{k_0}}\}^{\downarrow}, \{j_{i_{k_0+1}}\}^{\downarrow}}$ can be either 0 or 1. If it was 1, $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0}}\}^{\downarrow}$ would be the antiparticle of $\{j_{i_{k_0+1}}\}^{\downarrow}$. Because all particles are their own antiparticles, this would imply $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0}}\}^{\downarrow} = \{j_{i_{k_0+1}}\}^{\downarrow}$. However, this is not possible for $k_0 > 1$, because the associativity of fusion would then also imply that $\{j_{i_1}\}^{\downarrow}$ is the antiparticle (and coinciding with) $\{j_{i_2} \cdots j_{i_{k_0}} j_{i_{k_0+1}}\}^{\downarrow}$, i.e., $\{j_{i_1}\}^{\downarrow} = \{j_{i_2} \cdots j_{i_{k_0}} j_{i_{k_0+1}}\}^{\downarrow}$. Remembering that $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0}}\}, \{j_{i_2} j_{i_2} \cdots j_{i_{k_0}+1}\}$ do not split, and equating the quantum dimensions of the particles for these two identifications we have

$$\begin{aligned} d_{j_{i_1}} d_{j_{i_2}} \cdots d_{j_{i_{k_0}}} &= d_{j_{i_{k_0+1}}}, \\ d_{j_{i_1}} &= d_{j_{i_2}} \cdots d_{j_{i_{k_0}}} d_{j_{i_{k_0+1}}}. \end{aligned} \tag{B2}$$

For $k_0 > 1$, this contradicts the fact that all nontrivial particles in this theory have quantum dimensions $d > 1$. This rules out the possibility $\tilde{N}^{\varphi}_{\{j_{i_1} \cdots j_{i_{k_0}}\}^{\downarrow}, \{j_{i_{k_0+1}}\}^{\downarrow}} = 1$ and shows that $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0+1}}\}$ does not condense for $k_0 > 1$.

The case $k_0 = 1$ needs to be considered separately, as both lines in equation (B2) are identical in this case, and therefore do not lead to a contradiction. Assume that $\tilde{N}^{\varphi}_{\{j_{i_1}\}^{\downarrow}, \{j_{i_2}\}^{\downarrow}} = 1$. In the case $j_{i_1} \neq j_{i_2}$, we can rely on the following argument to disprove this assumption: as all anyons in $\mathrm{SO}(3)_k$ with k odd have distinct quantum dimension, it follows that the two anyons $\{j_{i_1}\}$ and $\{j_{i_2}\}$ restrict to distinct particles and in particular $\varphi \notin j_{i_1}^{\downarrow} \times j_{i_2}^{\downarrow}$—with equation (2) from the main text this implies that $\{j_{i_1} j_{i_2}\}$ neither splits nor condenses. In the case $j_{i_1} = j_{i_2} \equiv j$, define $\hat{j} \equiv \{j_{i_1} j_{i_2}\}$. We want to show that $\hat{j}$ does not condense. As there are no fermions in $\mathrm{SO}(3)_k$ with k odd, $\hat{j}$ can only be a boson if $\theta_j = 1$, i.e., if $\{j_{i_1}\}$ and $\{j_{i_2}\}$ are bosons. Our no-go theorem applies to all bosons $\{j_{i_1}\}$ and $\{j_{i_2}\}$ with zero mode sets $\mathcal{I}_{\{j_{i_1}\}}$ and $\mathcal{I}_{\{j_{i_2}\}}$. We can then use the set $\mathcal{I}_{\hat{j}} = \mathcal{I}_{\{j_{i_1}\}} \times \mathcal{I}_{\{j_{i_2}\}}$, containing the fusion product of any particle in $\mathcal{I}_{\{j_{i_1}\}}$ with any particle in $\mathcal{I}_{\{j_{i_2}\}}$, to prove that $\hat{j}$ cannot condense. To show that $\mathcal{I}_{\hat{j}}$ is a set of zero modes of $\hat{j}$, the main challenge is to show that the product of any two elements from $\mathcal{I}_{\hat{j}}$ cannot condense. The product of any two elements from $\mathcal{I}_{\hat{j}}$ is always of the form $\{j_{i_1} j_{i_2}\}$. We have shown that when $j_{i_1} \neq j_{i_2}$ such particles cannot condense. We therefore need only show that nontrivial particles of form $\{j_{i_1} j_{i_2}\}$ with $j_{i_1} = j_{i_2}$ both bosons cannot condense. In order to show they are not condensable, we can use the proof given for example (ii). For that, observe that the anyons $\hat{j}$ have the same fusion coefficients among themselves as the j anyons in $\mathrm{SO}(3)_k$ in example (ii) have, i.e., $N_{\hat{j},\hat{j}'}^{\hat{j}''} = N_{j,j'}^{j''}$, where $j, j', j'' \in \mathrm{SO}(3)_k$. Recall that conditions 1–3 from the definition of a set of zero modes only depend on the fusion coefficients $N_{j,j'}^{j''}$ and the information, which particles are bosons. Hence, conditions 1–3 are satisfied for $\mathcal{I}_{\hat{j}}$ whenever they are satisfied for $\mathcal{I}_j$ in example (ii). It remains to show that $\mathcal{I}_{\hat{j}}$ is of large enough quantum dimension to satisfy the fundamental inequality equation (4) from the main text. For $\hat{j}$, equation (4) from the main text takes the form

$$d_{\hat{j}} = d_j^2 < \left(\sum_{a\in\mathcal{I}_j} d_a\right)^2 = \sum_{\hat{a}\in\mathcal{I}_{\hat{j}}} d_{\hat{a}}. \tag{B3}$$

Upon taking the square root, this is equivalent to equations (14) and (15) from the main text, which were shown to hold in example (ii). Therefore the $\hat{j} = \{j_{i_1} j_{i_2}\}$ anyons do not condense and all $\{j_i\}$ have distinct restrictions.

We conclude that for any $k_0 \geqslant 1$ only $\tilde{N}^{\varphi}_{\{j_{i_1}\cdots j_{i_{k_0}}\}^{\downarrow},\{j_{i_{k_0+1}}\}^{\downarrow}} = 0$ is permitted and hence equation (B1) implies that $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0+1}}\}$ does not restrict to the identity φ, i.e., it does not condense. This proves the assumption 1 of the induction step for $k_0 + 1$.

To complete the induction step, we need to show that $\{j_{i_1} j_{i_2} \cdots j_{i_{k_0+1}}\}$ does not split. For that, consider equation (2) from the main text for $\{j_{i_1} \cdots j_{i_{k_0+1}}\}$ with itself and $t = \varphi$

$$\begin{aligned}\sum_r (n^r_{\{j_{i_1}\cdots j_{i_{k_0+1}}\}})^2 &= \sum_c N^c_{\{j_{i_1}\cdots j_{i_{k_0+1}}\},\{j_{i_1}\cdots j_{i_{k_0+1}}\}}\, n_c^{\varphi} \\ &= n_1^{\varphi} = 1.\end{aligned} \tag{B4}$$

We have used that none of the $\{j_{i_1} j_{i_2} \cdots j_{i_l}\}$ with $1 \leqslant l \leqslant k_0 + 1$ can restrict to the identity φ since they cannot condense. This implies none of $\{j_{i_1} \cdots j_{i_{k_0+1}}\}$ splits, which proves the assumption 2 of the induction step for $k_0 + 1$.

We have thus shown inductively that none of the particles (except for the vacuum) restricts to the vacuum in the N-layer theory $\mathrm{SO}(3)_k^{\otimes N}$. Thus, there is no condensate and with it no condensation in any number N of layers of $\mathrm{SO}(3)_k$ with k odd.

Appendix C. General constraints on boson condensation

In this section, we list lemmas that pose other general constraints on condensation transitions in TQFTs.

Lemma 2. *Suppose $S = \{a_1, \cdots, a_m\}$ is a collection of particles in a TQFT $\mathcal{A}$ with $a_i \times \bar{a}_i$ containing no bosons other than the identity—i.e., $n^t_{a_i} = \delta^t_{a_i^{\downarrow}}$ and a_i does not split. Moreover assume $a_i^{\downarrow} \neq a_j^{\downarrow}$ for $i \neq j$. Then if a boson B appears in the fusion of a_i and $\bar{a}_j$, $a_i \times \bar{a}_j = B + \cdots$ for any $i \neq j$, then B is not condensable.*

Proof. Using equation (2) from the main text for $a = a_i$, $b = \bar{a}_j$ and $t = \varphi$, we have $\delta_{ij} = \sum_t n^t_{a_i} n^t_{\bar{a}_j} = \sum_c n_c^{\varphi} N^c_{a_i \bar{a}_j}$. For $i \neq j$ we get $\sum_B n_B^{\varphi} N^B_{a_i \bar{a}_j} = 0$. So if boson B appears in $a_i \times \bar{a}_j$, we must have $n_B^{\varphi} = 0$, so that B is not condensable. □

Lemma 3. *Consider a TQFT $\mathcal{A}$ with no fusion multiplicity and just one boson B. If B is condensed then either B is abelian or $B^{\downarrow} = \varphi + r$ where r is a single anyon.*

Proof. As there is just a single boson, $B = \bar{B}$. Equation (2) from the main text implies

$$\sum_t n_B^t n_B^t = \sum_c n_c^{\varphi} N^c_{BB} = 1 + n_B^{\varphi} N^B_{BB}. \tag{C1}$$

Notice, however, that the left-hand side is greater or equal to $n_B^{\varphi} n_B^{\varphi}$. For condensation, this implies $n_B^{\varphi} = 1$, and tells us that $\sum_t n_B^t n_B^t = 1$ or 2. In the former case, $B^{\downarrow} = \varphi$. This implies $d_B = 1$, and so B is a quantum dimension 1 boson hence must have $N^a_{BB} = \delta_{a,1}$. In the latter case, B restricts to just two particles with multiplicity 1 each, so that $B^{\downarrow} = \varphi + r$. □

Lemma 4. *With the conditions of lemma 3, and assuming B has* $d_B > 1$, *condensation of B can only occur if* $N_{BB}^{a} N_{BB}^{b} \leqslant N_{ab}^{B}$ *for all anyons a and b of* $\mathcal{A}$.

Proof. Lemma 3 shows $B^{\downarrow} = \varphi + r$, where r a simple object. Consider $a \in \mathcal{A}$ where $a \neq 1, B$. Equation (2) from the main text for $b = B$ and $t = \varphi$ reads

$$\begin{aligned} n_a^r n_B^r &= n_a^r \\ &= N_{aB}^{B} n_B^{\varphi} \\ &= N_{aB}^{B}. \end{aligned} \tag{C2}$$

Consider now equation (2) from the main text for $b \neq 1, B$ and for $a \neq b$, which gives

$$\sum_t n_a^t n_b^t = N_{ab}^{B} \geqslant n_a^r n_b^r. \tag{C3}$$

Combining the a, B and b, B and a, b equations gives the inequality

$$N_{BB}^{b} N_{BB}^{a} \leqslant N_{ab}^{B}. \tag{C4}$$

□

References

[1] Tsui D C, Stormer H L and Gossard A C 1982 Two-dimensional magnetotransport in the extreme quantum limit *Phys. Rev. Lett.* **48** 1559
[2] Laughlin R B 1983 Anomalous quantum Hall effect: an incompressible quantum fluid with fractionally charged excitations *Phys. Rev. Lett.* **50** 1395
[3] Kitaev A Y 2006 Anyons in an exactly solved model and beyond *Ann. Phys., NY* **321** 2–111
[4] Kitaev A Y 2003 Fault-tolerant quantum computation by anyons *Ann. Phys.* **303** 2
[5] Wen X G, Wilczek F and Zee A 1989 Chiral spin states and superconductivity *Phys. Rev.* B **39** 11413
[6] Wen X G 1989 Vacuum degeneracy of chiral spin states in compactified space *Phys. Rev.* B **40** 7387
[7] Wen X G 1990 Topological orders in rigid states *Int. J. Mod. Phys.* B **4** 239
[8] Chen X, Gu Z C and Wen X G 2010 Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order *Phys. Rev.* B **82** 155138
[9] Wen X G 2016 A theory of 2 + 1D bosonic topological orders *Natl Sci. Rev.* **3** 68-106
[10] Wilczek F 1982 Quantum mechanics of fractional-spin particles *Phys. Rev. Lett.* **49** 957
[11] Bais F A and Romers J C 2012 The modular *S*-matrix as order parameter for topological phase transitions *New J. Phys.* **14** 035024
[12] He H, Moradi H and Wen X G 2014 Modular matrices as topological order parameter by a gauge-symmetrypreserved tensor renormalization approach *Phys. Rev.* B **90** 205114
[13] Moradi H and Wen X G 2015 Universal wave function overlap and universal topological data from generic gapped ground states *Phys. Rev. Lett.* **115** 036802
[14] Liu F, Wang Z, You Y Z and Wen X G 2013 Modular transformations and topological orders in two dimensions arXiv:1303.0829
[15] Kirillov A Jr and Ostrik V 2002 On q-analog of McKay correspondence and ADE classification of *sl* (2) conformal field theories *Adv. Math.* **171** 183
[16] Bais F A and Slingerland J K 2009 Condensate induced transitions between topologically ordered phases *Phys. Rev.* B **79** 045316
[17] Eliëns I S, Romers J C and Bais F A 2014 Diagrammatics for Bose condensation in anyon theories *Phys. Rev.* B **90** 195130
[18] You Y Z, Jian C M and Wen X G 2013 Synthetic topological degeneracy by anyon condensation *Phys. Rev.* B **87** 045106
[19] Barkeshli M and Wen X G 2010 Anyon condensation and continuous topological phase transitions in non-Abelian fractional quantum hall states *Phys. Rev. Lett.* **105** 216804
[20] Burnell F J, Simon S H and Slingerland J K 2012 Phase transitions in topological lattice models via topological symmetry breaking *New J. Phys.* **14** 015004
[21] Burnell F J, Simon Steven H and Slingerland J K 2011 Condensation of achiral simple currents in topological lattice models: Hamiltonian study of topological symmetry breaking *Phys. Rev.* B **84** 125434
[22] Hung L Y and Wan Y 2014 Symmetry-enriched phases obtained via pseudo anyon condensation *Int. J. Mod. Phys.* B **28** 1450172
[23] Kong L 2014 Anyon condensation and tensor categories *Nucl. Phys.* B **886** 436
[24] Hung L Y and Wan Y 2015 Generalized ADE classification of topological boundaries and anyon condensation *J. High Energy Phys.* JHEP07(2015)120
[25] Neupert T, He H, von Keyserlingk C W, Sierra G and Bernevig B A 2016 Boson condensation in topologically ordered quantum liquids *Phys. Rev.* B **93** 115103
[26] Bookera T and Davydov A 2012 Commutative algebras in Fibonacci categories *J. Algebra* **355** 176
[27] Witten E 1989 Quantum field theory and Jones polynomical *Commun. Math. Phys.* **121** 351
[28] Fröhlich J, Fuchs J, Runkel I and Schweigert C 2006 Correspondences of ribbon categories *Adv. Math.* **199** 192–329
[29] Levin M A and Wen X G 2005 String-net condensation: a physical mechanism for topological phases *Phys. Rev.* B **71** 045110
[30] Lan T and Wen X G 2014 Topological quasiparticles and the holographic bulk-edge relation in (2 + 1)-dimensional string-net models *Phys. Rev.* B **90** 115119
[31] Kong L and Wen X G 2014 Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions arXiv:1405.5858
[32] Kong L, Wen X G and Zheng H 2015 Boundary-bulk relation for topological orders as the functor mapping higher categories to their centers arXiv:1502.01690
[33] Bonderson P 2007 Non-abelian anyons and interferometry *PhD Thesis* http://thesis.library.caltech.edu/2447/2/thesis.pdfCalifornia Institute of Technology, Pasedena, California
[34] Bernevig B A and Neupert T 2015 Topological superconductors and category theory arXiv:1506.05805

[35] Barkeshli M, Bonderson P, Cheng M and Wang Z 2014 Symmetry defects, and gauging of topological phases arXiv:1410.4540
[36] Capelli A, Itzykson C and Zuber J-B 1987 Modular invariant partition functions in two-dimensions *Nucl. Phys.* B **280** 445
Capelli A, Itzykson C and Zuber J-B 1987 The ADE classification of minimal and $A_1^{(1)}$ conformal invariant theories *Commun. Math. Phys.* **113** 1
[37] Di Francesco F, Mathieu P and Sénéchal D 1997 *Conformal Field Theory* (Berlin: Springer)
[38] Levin M and Gu Z-C 2012 Braiding statistics approach to symmetry-protected topological phases *Phys. Rev.* B **86** 115109
[39] Chen X, Gu Z-C, Liu Z-X and Wen X-G 2013 Symmetry protected topological orders and the group cohomology of their symmetry group *Phys. Rev.* B **87** 155114